

Factorization theorems for dominated polynomials

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Abstract

In this note we prove that the factorization theorem for dominated polynomials given in [2] is equivalent to an alternative factorization scheme that uses classical linear techniques and a linearization process. However, this alternative scheme is shown not to be satisfactory until the equivalence is proved.

1 Introduction and notation

Let n be a positive integer and $1 \leq p < +\infty$. Generalizing the well-known Pietsch Factorization Theorem for absolutely p -summing linear operators, in [2] we prove the existence of a canonical prototype of an n -homogeneous p -dominated polynomial such that any other n -homogeneous p -dominated polynomial between Banach spaces is the composition of a restriction of it with some bounded linear operator. Using the linearization of homogeneous polynomials on projective symmetric tensor products, in this note we obtain an alternative factorization scheme, which is nonnatural at first glance (in a sense that will become clear in Remark 3). However, by relating the p -dominated norm of a p -dominated polynomial with the sup norms of the mappings involved in the factorization diagrams, we prove that this alternative approach is indeed equivalent to the factorization scheme provided in [2]. Although proofs are based mainly on the results in [2] and their proofs therein, the interest of this note is to clarify that the alternative factorization diagram, a priori non satisfactory, turns out to be equivalent to the previous one and that the construction in [2] comes into help to achieve this end.

From now on, E and F will denote (real or complex) Banach spaces, and B_{E^*} the closed unit ball of the dual space E^* of E . When endowed with

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the weak* topology, B_{E^*} is a compact space, and, as usual, $C(B_{E^*})$ means the space of all continuous functions on B_{E^*} endowed with the supremum norm $\|\cdot\|_\infty$. Let n be a positive integer and $1 \leq p < +\infty$. Given a regular Borel probability measure μ on B_{E^*} , let $j_{p/n}: C(B_{E^*}) \rightarrow L_{p/n}(\mu)$ be the canonical map.

Let us denote by $\mathcal{P}(^n E; F)$ the space of all continuous n -homogeneous polynomials from E into F endowed with the usual sup norm. For the theory of homogeneous polynomials we refer to the excellent monograph [3].

Consider the n -homogeneous polynomial $j_{p/n}^n: C(B_{E^*}) \rightarrow L_{p/n}(\mu)$ given by $j_{p/n}^n(f) = j_{p/n}(f^n)$. By e we mean the evaluation map from E into $C(B_{E^*})$ given by $e(x) = e_x$, where $e_x(x^*) = \langle x^*, x \rangle$ for any $x^* \in B_{E^*}$. The restriction of $j_{p/n}^n$ to $e(E)$ will be denoted $(j_{p/n}^e)^n$, so $(j_{p/n}^e)^n: e(E) \rightarrow E^{p/n}$ where $E^{p/n} := (j_{p/n}^e)^n \circ e(E)$.

A continuous n -homogeneous polynomial $P: E \rightarrow F$ is said to be p -dominated if there is a constant $C > 0$ such that

$$\left(\sum_{j=1}^m \|P(x_j)\|^{p/n} \right)^{n/p} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{j=1}^m |\langle x^*, x_j \rangle|^p \right)^{n/p}$$

for any positive integer m and any finite sequence of vectors $x_1, \dots, x_m \in E$. The infimum of the constants C for which the above inequality holds is denoted by $\|P\|_{d,p}$. It is well known that $\|\cdot\|_{d,p}$ makes the set $\mathcal{P}_{d,p}(^n E; F)$ of all p -dominated n -homogeneous polynomials from E into F a Banach space if $p \geq n$ and a complete p/n -normed space if $p < n$ (see e.g. [1]). If $n = 1$ we recover the classical ideal of absolutely p -summing linear operators.

2 Alternative factorization scheme

In order to describe the factorization theorem for dominated polynomials proved in [2] and get the alternative approach we need to introduce some notation. By $\otimes_{\pi_s}^{n,s} E$ we denote the n -fold symmetric tensor product of E endowed with the projective s -tensor norm π_s (see [4] for definitions and main properties). As usual, $\hat{\otimes}_{\pi_s}^{n,s} E$ denotes the completed space. Let $\delta_n^E: E \rightarrow \otimes_{\pi_s}^{n,s} E$ be the diagonalization map given by $\delta_n^E(x) = x \otimes \dots \otimes x$. Given $P \in \mathcal{P}(^n E; F)$, $P^L: \otimes_{\pi_s}^{n,s} E \rightarrow F$ is the linearization of P , that is the unique continuous linear map from $\otimes_{\pi_s}^{n,s} E$ to F fulfilling $P^L \circ \delta_n^E = P$. It is well known that $\|P^L\| = \|P\|$.

According to [2], there is an injective linear operator $\delta: \otimes_{\pi_s}^{n,s} E \rightarrow C(B_{E^*})$ such that $\delta(x \otimes \dots \otimes x)(x^*) = \langle x^*, x \rangle^n$ for any $x \in E$ and $x^* \in B_{E^*}$, and

$j_{p/n} \circ \delta(\otimes_{\pi_s}^{n,s} E) = (j_{p/n}^e)^n \circ e(E) = E^{p/n}$. So, $E^{p/n}$ is a linear subspace of $L_{p/n}(\mu)$. Moreover, in [2] a norm $\pi_{p/n}$ is defined on $E^{p/n}$ by:

$$\pi_{p/n}((j_{p/n} \circ \delta)(\theta)) := \inf \left\{ \sum_{i=1}^m |\lambda_i| \| (j_{p/n} \circ \delta)(x_i \otimes \cdots \otimes x_i) \|_{L_{p/n}} \right\},$$

where the infimum is taken over all representations of $\theta \in \otimes_{\pi_s}^{n,s} E$ of the form $\theta = \sum_{i=1}^m \lambda_i x_i \otimes \cdots \otimes x_i$ with $m \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$ and $x_i \in E$.

Let us recall the factorization theorem for dominated polynomials that appears in [2, Theorem 4.6]:

Theorem 1 [2, Theorem 4.6] *Let $P \in \mathcal{P}(^n E; F)$ and $1 \leq p < +\infty$. Then P is p -dominated if and only if there is a regular Borel probability measure μ on B_{E^*} with the weak* topology and a continuous linear operator $u: (E^{p/n}, \pi_{p/n}) \longrightarrow F$ such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ e \downarrow & & \uparrow u \\ e(E) & \xrightarrow{(j_{p/n}^e)^n} & E^{p/n} \\ \downarrow & & \downarrow \\ C(B_{E^*}) & \xrightarrow{(j_{p/n})^n} & L_{p/n}(\mu) \end{array}$$

Moreover, $\|u\| = \|P\|_{d,p}$.

Although the equality $\|u\| = \|P\|_{d,p}$ does not appear in the statement of [2, Theorem 4.6], it can be obtained from [2] as follows:

From the proof of [2, Theorem 4.6] we have that $\|u\| \leq \|P\|_{d,p}$. In [2, Lemma 4.5] it is proved that $(j_{p/n}^e)^n$ is p -dominated and that for every $x_1, \dots, x_m \in E$,

$$\left(\sum_{i=1}^m \pi_{p/n} \left((j_{p/n}^e)^n(e(x_i)) \right)^{p/n} \right)^{n/p} \leq \sup_{\psi \in B_{C(B_{E^*})^*}} \left(\sum_{j=1}^m |\langle \psi, e(x_j) \rangle|^p \right)^{n/p}.$$

It follows that $\|(j_{p/n}^e)^n\|_{d,p} \leq 1$. Since $P = u \circ (j_{p/n}^e)^n \circ e$, using the ideal property [2, Proposition 2.2] we get

$$\|P\|_{d,p} = \|u \circ (j_{p/n}^e)^n \circ e\|_{d,p} \leq \|u\| \|(j_{p/n}^e)^n\|_{d,p} \|e\|^n = \|u\|.$$

By combining the classical linear factorization techniques with a linearization process we get a different approach to the factorization scheme given in Theorem 1. Naturally, j_p^e denotes the restriction of j_p to $e(E)$ onto its range.

Proposition 2 *Let $P \in \mathcal{P}^n(E; F)$ and $1 \leq p < +\infty$. Then P is p -dominated if and only if there is a regular Borel probability measure μ on B_{E^*} with the weak* topology, a closed subspace $G_{E,p}$ of $L_p(\mu)$ containing $j_p(e(E))$ and a continuous linear operator $v: \otimes_{\pi_s}^{n,s} G_{E,p} \longrightarrow F$ such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ e \downarrow & & \uparrow v \\ e(E) & \xrightarrow{\delta_n^{G_{E,p}} \circ j_p^e} & \otimes_{\pi_s}^{n,s} G_{E,p} \end{array}$$

Moreover, $\|v\| = \|P\|_{d,p}$.

PROOF: Assume that P is p -dominated. Adapting to polynomials the proof of the linear case (the multilinear case can be found in [5, Teorema 3.6]) or combining Pietsch Factorization Theorem with [2, Proposition 3.4], we can conclude that there is a regular Borel probability measure μ on B_{E^*} and a continuous n -homogeneous polynomial Q from $G_{E,p} := \overline{j_p^e \circ e(E)} \subset L_p(\mu)$ to F such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ e \downarrow & & \uparrow Q \\ e(E) & \xrightarrow{j_p^e} & G_{E,p} \\ \downarrow & & \downarrow \\ C(B_{E^*}) & \xrightarrow{j_p} & L_p(\mu) \end{array}$$

that is, $Q \circ j_p^e \circ e = P$. Moreover, $\|Q\| \leq \|P\|_{p,d}$. Defining $v := Q^L$ as the linearization of Q on $\otimes_{\pi_s}^{n,s} G_{E,p}$ we obtain the desired commutative diagram.

As to the converse, by [2, Proposition 3.4] we know that $\delta_n^{G_{E,p}} \circ j_p^e$ is p -dominated, so $P = v \circ (\delta_n^{G_{E,p}} \circ j_p^e) \circ e$ is p -dominated as well.

Denoting by $\pi_p(j_p^e)$ the p -summing norm of j_p^e , the following computation completes the proof:

$$\|P\|_{d,p} = \|Q^L \circ \delta_n^{G_{E,p}} \circ j_p^e \circ e\|_{d,p} \leq \|Q^L\| \|\delta_n^{G_{E,p}} \circ j_p^e\|_{d,p} \|e\|^n$$

$$\leq \|Q^L\| \|\delta_n^{G_{E,p}}\| (\pi_p(j_p^e))^n \leq \|Q^L\| = \|Q\| \leq \|P\|_{d,p}.$$

■

Remark 3 (a) It is worth mentioning that the intermediate factorization scheme $P = Q \circ j_p^e \circ e$ in the proof above provides the factorization of p -dominated polynomials through a canonical absolutely p -summing *linear operator*.

(b) Observe that the n -homogeneous polynomial $\delta_n^{G_{E,p}} \circ j_p^e$ is p -dominated by [2, Proposition 3.4]. Theorem 2 assures that, like the polynomial $(j_{p/n}^e)^n$ of Theorem 1, $\delta_n^{G_{E,p}} \circ j_p^e$ is a canonical prototype of a p -dominated polynomial through which every p -dominated polynomial factors. Let us explain why Theorem 2 is a somewhat nonnatural generalization of Pietsch Factorization Theorem to p -dominated polynomials. Remember that Pietsch Factorization Theorem gives the factorization of a p -summing linear operator $u: E \rightarrow F$ through a canonical prototype of a p -summing linear operator from a subspace of $C(B_{E^*})$ into some subspace of $L_p(\mu)$, where μ is some regular Borel probability measure on B_{E^*} . Observe that this is exactly what happens for polynomials in Theorem 1, whereas in Theorem 2 we do not know (up to now) if $\hat{\otimes}_{\pi_s}^{n,s} G_{E,p}$ is a subspace of $L_{q_n}(\mu)$ for some number q_n with $q_1 = p$ and some regular Borel probability measure μ on (B_{E^*}, w^*) .

We finish the note by proving that the completion $\overline{E}^{p/n}$ of the space $E^{p/n}$ of Theorem 1 is isometrically isomorphic to the completion $\hat{\otimes}_{\pi_s}^{n,s} G_{E,p}$ of the space $\hat{\otimes}_{\pi_s}^{n,s} G_{E,p}$ of Proposition 2. This shows that the two factorization schemes are equivalent, a fact which reinforces the role of Theorem 1 as a generalization of Pietsch Factorization Theorem to dominated polynomials.

Theorem 4 *The spaces $(\overline{E}^{p/n}, \pi_{p/n})$ and $\hat{\otimes}_{\pi_s}^{n,s} G_{E,p}$ are isometrically isomorphic.*

PROOF: It suffices to show that $(\overline{E}^{p/n}, \pi_{p/n})$ and $\hat{\otimes}_{\pi_s}^{n,s} j_p^e \circ e(E)$ are isometrically isomorphic. The proof is based on the proof of [2, Proposition 4.2]. Consider $T := \hat{\otimes}_{\pi_s}^{n,s} j_p^e \circ e$ the symmetric n -fold tensor product of the linear operator $j_p^e \circ e$. By [4, 1.7], $T: \hat{\otimes}_{\pi_s}^{n,s} E \rightarrow \hat{\otimes}_{\pi_s}^{n,s} j_p^e \circ e(E)$ is a linear operator and satisfies

$$T(x \otimes \cdots \otimes x) = j_p^e(e(x)) \otimes \cdots \otimes j_p^e(e(x)) \text{ for every } x \in E.$$

On the other hand, T is also injective and

$$\pi_s(T(\theta)) = \pi_{p/n}(j_{p/n} \circ \delta(\theta)) \quad (1)$$

for all $\theta \in \otimes^{n,s} E$. Then $T: \otimes_{\pi_s}^{n,s} E \longrightarrow \otimes_{\pi_s}^{n,s} j_p^e \circ e(E)$ is a linear bijection. Hence $j_{p/n} \circ \delta \circ T^{-1}: \otimes_{\pi_s}^{n,s} j_p^e \circ e(E) \longrightarrow E^{p/n}$ is a linear bijection, and by (1) this map is an isometry. Taking completions we get the desired conclusion. ■

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